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Periodic and subharmonic solutions of a class of superquadratic Hamiltonian systems ☆

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Abstract

Periodic solutions and infinitely distinct subharmonic solutions are obtained for a class of nonconvex and nonautonomous superquadratic Hamiltonian systems $\dot{z} = \mathcal{J} H_z(z,t)$ by using the minimax methods in critical point theory.

Keywords: Periodic solutions; Subharmonic solutions; Superquadratic Hamiltonian systems; Minimax method; Truncating argument; Critical point

1. Introduction and main results

Consider the nonautonomous Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(z,t), \tag{1.1}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

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is the standard symplectic matrix, $H : \mathcal{R}^{2n} \times \mathcal{R} \to \mathcal{R}$ is a C^1 function, and T-periodic in the second variable. A usual hypothesis under which interesting results for (1.1) can be obtained is

(C) There exist $\mu > 2$ and $r_0 > 0$ such that

$$\frac{1}{\mu} H_z(z,t) \cdot z \geqslant H(z,t) > 0 \quad \forall z \in \mathcal{R}^{2n},\ |z| \geqslant r_0,\ \forall t \in \mathcal{R}.$$

Here, and in what follows, we denote by $\cdot$ the usual inner product in $\mathcal{C}^m$ and by $|\cdot|$ the corresponding norm. A Hamiltonian satisfying (C) is called superquadratic. There are several papers dealing with the existence of T-periodic solutions of (1.1) under condition (C) (see, e.g., [1–10,17]).

A particular case of Eq. (1.1) is the so-called second order Hamiltonian system

$$\ddot{q} + \nabla V(q,t) = 0, \tag{1.2}$$

where $V : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}$ is a C^1 function, and T-periodic in the second variable. If we put

$$H(p,q,t) = \frac{1}{2}|p|^2 + V(q,t), \tag{1.3}$$

(1.2) is equivalent to (1.1). Several results for the existence of T-periodic solutions for (1.2) have been obtained. A usual hypothesis to consider on V is

(2C) There exist $\mu > 2$ and $r_0 > 0$ such that

$$\frac{1}{\mu} \nabla V(q,t) \cdot q \geqslant V(q,t) > 0 \quad \forall q \in \mathcal{R}^n,\ |q| \geqslant r_0,\ \forall t \in \mathcal{R}.$$

See, for example, [8,11–13] and others. We observe that H does not satisfy (C) if V satisfies (2C) and H is defined by (1.3).

In 1993, Felmer [1] extend some existence result for (1.1) where the Hamiltonian satisfies a superquadratic condition that include simultaneously (C) and (2C). In [1], Felmer gave the following condition:

(3C) There exist $\alpha > 1$, $\beta > 1$, $1/\alpha + 1/\beta < 1$ and $r_0 > 0$ such that

$$\frac{1}{\alpha} H_p(p,q,t) \cdot p + \frac{1}{\beta} H_q(p,q,t) \cdot q \geqslant H(p,q,t) > 0$$
$$\forall z = (p,q) \in \mathcal{R}^n \times \mathcal{R}^n,\ |z| \geqslant r_0,\ \forall t \in \mathcal{R}.$$

H obviously satisfies (3C) when it satisfies (C). It is easy to obtain that when V satisfies (2C) and H is defined by (1.3), H satisfies (3C) too.

Felmer [1] had well solved the unifying of superquadratic conditions of Eqs. (1.1) and (1.2), but the result of Felmer in [1] does not include the corresponding result in [7] as a special case. A natural question is whether there exists a result which contains the corresponding results in [1] and [7] as a special case.

Motivated by [1] and [7], we give this question a positive answer by the minimax methods in the critical point theory and obtain a result (see Theorem 1.1) unifies and generalizes Theorem 0.1 in [1] and Theorem 2.49 in [7]. In this paper, we consider a Hamiltonian satisfying the following hypothesis:

(H0) H is of class C^1, and

$$H(z,t+T)=H(z,t) \quad \forall z\in\mathcal{R}^{2n},\ \forall t\in\mathcal{R},$$

(H1) $H(z,t)\geqslant 0\ \forall t\in\mathcal{R}, \forall z\in\mathcal{R}^{2n}$,

(H2) $\exists\alpha>1$, $\beta>1$, $1/\alpha+1/\beta<1$ and $r_0>0$ such that

$$\frac{1}{\alpha}H_p(p,q,t)\cdot p+\frac{1}{\beta}H_q(p,q,t)\cdot q\geqslant H(p,q,t)>0$$
$$\forall z=(p,q)\in\mathcal{R}^{2n},\ |z|\geqslant r_0,\ \forall t\in\mathcal{R},$$

(H3) $$\lim_{|(p,q)|\to 0}\frac{H(p,q,t)}{|p|^{1+\alpha/\beta}+|q|^{1+\beta/\alpha}}=0,$$

(H4) $\exists a>0$ and $b\geqslant 0$ such that

$$\bigl|H_z(p,q,t)\bigr|\leqslant a\left(\frac{1}{\alpha}H_p(p,q,t)\cdot p+\frac{1}{\beta}H_q(p,q,t)\cdot q\right)+b$$
$$\forall z=(p,q)\in\mathcal{R}^{2n},\ \forall t\in\mathcal{R}.$$

We obtain the following theorem.

Theorem 1.1. *Suppose that H satisfies* (H0)–(H4)*. Then Eq.* (1.1) *has a nonzero T-periodic solution.*

Remark 1.1. It is easy to obtain that our Theorem 1.1 unifies and generalizes Theorem 0.1 in [1], Theorem 2.49 in [7] and Corollary 1.5 in [5]. Even if it is in the autonomous case, there are functions H satisfying our Theorem 1.1 and not satisfying the corresponding results in [1,7] and others. In fact, for $\alpha>1$, $\beta>1$ satisfying $1/\alpha+1/\beta<1$, let

$$H(z)=a_1\bigl(|p|^{1+\alpha/\beta}+|q|^{1+\beta/\alpha}\bigr)^{\gamma_1}+a_2\bigl(|p|^{1+\alpha/\beta}+|q|^{1+\beta/\alpha}\bigr)^{\gamma_2}, \tag{1.4}$$

where $a_1>0$, $a_2>0$, $1<\gamma_1<\alpha\beta/(\alpha+\beta)<\gamma_2$. Then H satisfies the conditions of our Theorem 1.1, but does not satisfy the corresponding theorems in [1–10,17].

In the following, we search for kT periodic solutions (called subharmonics) of (1.1) and (1.2). Several results for the existence of subharmonic solutions for (1.1) have been obtained when H is convex (see [5,14–16,18,19]). There are also several papers dealing with the existence of subharmonic solutions of (1.1) when H is nonconvex (see [5,10,20–22]). In [5,10,21–23], subharmonic solutions are obtained for superquadratic Hamiltonian systems. In [5,20], subharmonic solutions are obtained for subquadratic Hamiltonian systems. Many results for the existence of subharmonic solutions for (1.2) have been obtained (see [5,14,18,24–31]). Here, we give the following theorem.

Theorem 1.2. *Suppose that H satisfies* (H0)–(H4). *Then* (1.1) *possesses infinitely distinct subharmonic solutions.*

Remark 1.2. Theorem 1.2 generalizes Theorem 1.36 of [5] in the case that the quadratic form vanishes. It s easy to obtain that the functions H in (1.4) satisfy the conditions of our Theorem 1.2, but do not satisfy that of the corresponding theorems in [5,14–16,18–23].

2. Proofs of the main results

We study the existence of T-periodic solutions of (1.1) from the variational point of view. Taking $\omega = 2\pi/T$, let $E := W^{1/2,2}([0,T],\mathcal{R}^{2n})$ be the Sobolev's space of T-periodic $\mathcal{R}^{2n}$-valued functions

$$z(t) = \sum_{j\in\mathcal{Z}} a_j e^{\omega ijt}, \quad a_{-j} = \bar{a}_j \in \mathcal{C}^{2n},$$

such that

$$\|z\|^2 = \frac{T}{2}\sum_{j\in\mathcal{Z}-\{0\}} |j||a_j|^2 + T|a_0|^2 < \infty.$$

An inner product in E is defined by

$$\langle z,\eta\rangle = \frac{T}{2}\sum_{j\in\mathcal{Z}-\{0\}} |j|a_j\cdot\bar{b}_j + Ta_0\cdot\bar{b}_0.$$

It is well known that E compactly embedded in $L^\gamma([0,T],\mathcal{R}^{2n}) \equiv L^\gamma$ for $\gamma\in[1,+\infty)$ and as a consequence there exists a constant $C>0$ such that

$$\|z\|_\gamma \leqslant C\|z\| \quad \forall z\in E \tag{2.1}$$

for

$$\gamma = 1,\ \alpha,\ \beta,\ 1+\frac{\alpha}{\beta},\ 1+\frac{\beta}{\alpha}.$$

Here and in what follows $\|\cdot\|_\gamma$ denotes the usual norm in L^γ. For $z=(p,q)$ and $\eta = (\phi,\psi)$ in E and smooth we define

$$B(z,\eta) = \int_0^T (p\cdot\dot{\psi} + \phi\cdot\dot{q})\,dt, \qquad A(z) = \frac{1}{2}B(z,z). \tag{2.2}$$

Both A and B can be extended continuously to the whole space E, and the bilinear form B induces a linear, bounded, selfadjoint operator $L: E\to E$ defined by

$$B(z,\eta) = \langle Lz,\eta\rangle \quad \forall z,\eta\in E. \tag{2.3}$$

In E we can consider the splitting

$$E = E_+ \oplus E_- \oplus E_0, \tag{2.4}$$

where

$$\begin{aligned} E_+ = \overline{\operatorname{span}}^E \bigl\{ & \sin(\omega jt)e_k - \cos(\omega jt)e_{k+n}, \\ & \cos(\omega jt)e_k + \sin(\omega jt)e_{k+n} \mid j \in \mathcal{N},\ 1 \leqslant k \leqslant n \bigr\}, \\ E_- = \overline{\operatorname{span}}^E \bigl\{ & \sin(\omega jt)e_k + \cos(\omega jt)e_{k+n}, \\ & \cos(\omega jt)e_k - \sin(\omega jt)e_{k+n} \mid j \in \mathcal{N},\ 1 \leqslant k \leqslant n \bigr\}, \end{aligned}$$

and

$$E_0 = \operatorname{span}\{e_1, \dots, e_{2n}\}.$$

Here $\{e_1, \dots, e_{2n}\}$ is the canonical basis for $\mathcal{R}^{2n}$. We observe that A is positive on E_+, negative on E_-, and it vanishes on E_0. Actually E_+, E_-, and E_0 are the positive, negative, and null eigenspace of the linear operator L, respectively. On the space E we can see that

$$\|z\|^2 = A(z_+) - A(z_-) + T|z_0|^2, \tag{2.5}$$

where $z = z_+ + z_- + z_0$ with $z_+ \in E_+$, $z_- \in E_-$ and $z_0 \in E_0$.

Formally the T-periodic solutions of (1.1) are the critical points of the functional

$$J(z) = A(z) - \int_0^T H(z,t)\,dt.$$

However since we do not have an adequate control on the growth of H, J is not needed to be well defined. To overcome this difficulty we use a truncating argument introduced by Rabinowitz [7]. Let $K \geqslant 1$ be a constant and $\chi_K \in C^\infty(\mathcal{R}, \mathcal{R})$ such that $\chi_K(s) \equiv 1$ if $s \leqslant K$, $\chi_K(s) \equiv 0$ if $s \geqslant K+1$, and $\chi_K'(s) < 0$ for $s \in (K, K+1)$. We define

$$H_K(z,t) = \chi_K\bigl(|z|\bigr)H(z,t) + \bigl(1 - \chi_K\bigl(|z|\bigr)\bigr)r_K\bigl(|p|^\alpha + |q|^\beta\bigr),$$

where

$$r_K = \max\left\{ \frac{H(z,t)}{|p|^\alpha + |q|^\beta} \colon K \leqslant |z| \leqslant K+1,\ t \in \mathcal{R} \right\}.$$

An easy computation shows that H_K satisfies the same hypothesis as H. Moreover, hypothesis (H2) and (H4) are satisfied with constants independent of K. With this change the functional

$$J_K(z) = A(z) - \int_0^T H_K(z,t)\,dt$$

is well defined on E and it is of class C^1. Next we show a lemma regarding the growth of H as a consequence of hypothesis (H0) and (H2) and hence also valid for H_K.

Lemma 2.1 [1]. *If H satisfies* (H0) *and* (H2), *there exist constants $c_1 \geqslant 0$ and $c_2 \geqslant 0$ such that*

$$H(p,q,t) \geqslant c_1\bigl(|p|^\alpha + |q|^\beta\bigr) - c_2 \quad \forall (p,q) \in \mathcal{R}^{2n},\ \forall t \in \mathcal{R}.$$

For needs of our proofs, we introduce the following abstract theorem due to Felmer [1].

We consider a Hilbert space E with inner product $\langle \cdot \rangle$ and norm $\|\cdot\|$. We assume that E has a splitting $E = X \oplus Y$, where the subspace X and Y are not necessarily orthogonal and both of them can be infinite dimensional. Let $I : E \to R$ be a functional having the structure

$$I(z) = \langle Lz, z\rangle + b(z).$$

(I1) $L : E \to E$ is a linear, bounded, selfadjoint operator,
(I2) b' is compact,

there are two linear bounded, invertible operators $B_1, B_2 : E \to E$ satisfying

(I3) if $v \in \mathcal{R}_+$, the linear operator
$$\hat{B}(v) = P_X B_1^{-1} \exp(vL) B_2 : X \to X$$
is invertible. Here P_X denotes the projection of E onto X inducing by the splitting $E = X \oplus Y$, and $\mathcal{R}_+$ is a set of nonnegative real numbers.

Let $\rho > 0$ and define

$$S = \big\{B_1 z \mid \|z\| = \rho,\ z \in Y\big\}. \tag{2.6}$$

For $z_+ \in Y$, $z_+ \neq 0$, $\sigma > \rho/\|B_1^{-1} B_2 z_+\|$, and $M > 0$, we define

$$Q = \big\{B_2(sz_+ + z) \mid 0 \leqslant s \leqslant \sigma,\ \|z\| \leqslant M,\ z \in X\big\}. \tag{2.7}$$

We define ∂Q as the boundary of Q relative to the subspace $\{B_2(sz_+ + z) \mid s \in \mathcal{R},\ z \in X\}$. Let us consider the class of functions

$$\Gamma = \big\{h \in C\big(E \times [0,1], E\big) \mid h \text{ satisfies } \Gamma_1,\ \Gamma_2,\ \text{and } \Gamma_3\big\},$$

where

(Γ_1) $h(z,t) = \exp(v(z,t)L)z + \tilde{K}(z,t)$, where $v : E \times [0,1] \to \mathcal{R}^+$ is continuous and transforms bounded sets into bounded sets, and $\tilde{K} : E \times [0,1] \to E$ is compact,
(Γ_2) $h(z,t) = z\ \forall z \in \partial Q$,
(Γ_3) $h(z,0) = z\ \forall z \in Q$.

Theorem 2.2 [1]. *Let $I : E \to R$ be a C^1 functional satisfying the Palais–Smale condition and* (I1)–(I3). *Furthermore assume that there is a constant $\delta > 0$ such that*

(i) $I(z) \geqslant \delta\ \forall z \in S$,
(ii) $I(z) \leqslant 0\ \forall z \in \partial Q$.

Then I possesses a critical point with critical value $d \geqslant \delta$ characterized by

$$d = \inf_{h\in\Gamma} \sup_{z\in Q} I\big(h(z,1)\big).$$

Before proving our theorems, we obtained some useful facts regarding the subspaces E_+ and E_- and some estimates. Given a function $z \in L^\gamma$, $\gamma \geqslant 1$, with Fourier series given by

$$z = \sum_{k \geqslant 1} a_k \cos \omega kt + b_k \sin \omega kt,$$

with $a_k, b_k \in \mathcal{R}^n$, we define the conjugate of z by

$$\bar{z} = \sum_{k \geqslant 1} b_k \cos \omega kt - a_k \sin \omega kt.$$

We note that $\bar{\bar{z}} = -z \ \forall z \in L^\gamma$.

Lemma 2.3 [1]. *An element $z = (p, q) \in E$ belongs to E_+ (respectively E_-) if and only if $\bar{p} = -q$ (respectively $\bar{p} = q$).*

Lemma 2.4 [1]. *For every $z = (p, q) \in E_+$ or E_-, it holds that*

$$\|p\| = \|q\| = \frac{1}{\sqrt{2}} \|z\|.$$

Lemma 2.5 [1]. *Let $P_+ : E \to E$ (respectively $P_- : E \to E$) be projections induced by the splitting $E = E_+ \oplus E_- \oplus E_0$. If $z = (p, q) + z_0$, $z_0 \in E_0$, one has*

$$P_+(z) = \frac{1}{2}(p + \bar{q}, q - \bar{p}) \quad \left(\textit{respectively } P_-(z) = \frac{1}{2}(p - \bar{q}, q + \bar{p}) \right).$$

We define the operator B_1 and B_2 and the splitting E. Let $X = E_- + E_0$ and $Y = E_+$. We define $B_1 : E \to E$ by

$$B_1(z) = B_1((p, q)) = (\rho^{\beta-1} p, \rho^{\alpha-1} q), \tag{2.8}$$

and $B_2 : E \to E$ by

$$B_2(z + z_0) = B_2\big((p, q) + z_0\big) = (\sigma^{\beta-1} p, \sigma^{\alpha-1} q) + z_0, \tag{2.9}$$

where $z_0 \in E_0$ and $z = (p, q) \in E_- + E_+$, and the constants ρ and δ will be defined in Lemmas 2.7 and 2.8. Certainly B_1 and B_2 are bounded linear operators and both of them are invertible. From (2.6) and (2.8), we obtain

$$S = \big\{(\rho^{\beta-1} p, \rho^{\alpha-1} q) \mid \big\|(p, q)\big\| = \rho,\ (p, q) \in E_+\big\}. \tag{2.10}$$

By (2.7) and (2.9), we have

$$\begin{aligned} Q = \big\{ & s(\sigma^{\beta-1} p_+, \sigma^{\alpha-1} q_+) + (\sigma^{\beta-1} p, \sigma^{\alpha-1} q) + z_0 \mid \\ & 0 \leqslant s \leqslant \sigma,\ 0 \leqslant \big\|(p, q) + z_0\big\| \leqslant M,\ (p, q) \in E_-,\ z_0 \in E_0\big\}, \end{aligned} \tag{2.11}$$

where $z_+ = (p_+, q_+) \in E_+$ is a fixed eigenvector of L associated to an eigenvalue $\lambda > 0$ and $\|z_+\| = 1$. In what follows we denote by ∂Q the boundary of Q relative to the subspace

$$\big\{s(\sigma^{\beta-1} p_+, \sigma^{\alpha-1} q_+) + (\sigma^{\beta-1} p, \sigma^{\alpha-1} q) + z_0 \mid s \in \mathcal{R},\ (p, q) \in E_-,\ z_0 \in E_0\big\}.$$

Lemma 2.6 [1]. *The functional J_K satisfies the Palais–Smale condition.*

Lemma 2.7. *There exist $\rho > 0$ such that* (i) *is satisfied for J_K when S is defined by* (2.10).

Proof. From hypothesis (H3) and the form of H_K we have for each $\varepsilon > 0$,

$$H_K(p,q,t) \leqslant \varepsilon\big(|p|^{1+\alpha/\beta} + |q|^{1+\beta/\alpha}\big) + c_3\big(|p|^\alpha + |q|^\beta\big), \tag{2.12}$$

where $c_3 = c_3(\varepsilon, K) > 0$. Let $(p,q) \in E_+$ and take $z = (\rho^{\beta-1}p, \rho^{\alpha-1}q)$ for some $\rho > 0$. Then from (2.12), we have

$$\begin{aligned} J_K(z) \geqslant{}& \rho^{\alpha+\beta-2}\big\|(p,q)\big\|^2 - \varepsilon\big(\rho^{(\beta-1)(1+\alpha/\beta)}\|p\|_{1+\alpha/\beta}^{1+\alpha/\beta} + \rho^{(\alpha-1)(1+\beta/\alpha)}\|q\|_{1+\beta/\alpha}^{1+\beta/\alpha}\big) \\ & - c_3\big(\rho^{(\beta-1)\alpha}\|p\|_\alpha^\alpha + \rho^{(\alpha-1)\beta}\|q\|_\beta^\beta\big). \end{aligned} \tag{2.13}$$

From (2.13) and (2.1) we obtain

$$\begin{aligned} J_K(z) \geqslant{}& \rho^{\alpha+\beta-2}\big\|(p,q)\big\|^2 - \varepsilon c_4\big(\rho^{(\beta-1)(1+\alpha/\beta)}\|p\|^{1+\alpha/\beta} \\ & + \rho^{(\alpha-1)(1+\beta/\alpha)}\|q\|^{1+\beta/\alpha}\big) - c_5\big(\rho^{(\beta-1)\alpha}\|p\|^\alpha + \rho^{(\alpha-1)\beta}\|q\|^\beta\big), \end{aligned} \tag{2.14}$$

where $c_4 = \max\{C^{1+\alpha/\beta}, C^{1+\beta/\alpha}\}$, $c_5 = c_3\max\{C^\alpha, C^\beta\}$. If we consider $\|(p,q)\| = \rho$, from Lemma 2.4 and (2.14) one obtains

$$\begin{aligned} J_K(z) \geqslant{}& \rho^{\alpha+\beta} - \varepsilon c_4\rho^{\alpha+\beta}\left(\left(\frac{1}{\sqrt{2}}\right)^{1+\alpha/\beta} + \left(\frac{1}{\sqrt{2}}\right)^{1+\beta/\alpha}\right) \\ & - c_5\rho^{\alpha\beta}\left(\left(\frac{1}{\sqrt{2}}\right)^\alpha + \left(\frac{1}{\sqrt{2}}\right)^\beta\right). \end{aligned} \tag{2.15}$$

Noticing $1/\sqrt{2} < 1$, then we have from (2.15),

$$J_K(z) \geqslant (1 - 2\varepsilon c_4)\rho^{\alpha+\beta} - 2c_5\rho^{\alpha\beta}.$$

Taking $\varepsilon = 1/(4c_4)$, and then, since α and β satisfy $1/\alpha + 1/\beta < 1$, there exist $\rho > 0$ and $\delta > 0$ such that

$$J_K(z) \geqslant \delta > 0$$

and this inequality holds for $z \in S$, according to the definition of S. □

Lemma 2.8. *There are constants $\sigma > 0$ and $M > 0$ such that J_K satisfies* (ii) *for Q defined by* (2.11).

Proof. For $s \in \mathcal{R}^+$, $(p,q) \in E_-$, and $z_0 = (p_0, q_0) \in E_0$ we take

$$z = s(\sigma^{\beta-1}p_+, \sigma^{\alpha-1}q_+) + (\sigma^{\beta-1}p, \sigma^{\alpha-1}q) + z_0.$$

Then we have

$$
\begin{aligned}
A(z) = \frac{1}{2}B(z,z) &= \int_0^T \left(s\sigma^{\beta-1}p_+ + \sigma^{\beta-1}p + p_0\right)\cdot\left(s\sigma^{\alpha-1}\dot q_+ + \sigma^{\alpha-1}\dot q\right)dt \\
&= \int_0^T \sigma^{\alpha+\beta-2}\left(s^2 p_+\cdot\dot q_+ + p\cdot\dot q\right)dt \\
&= \sigma^{\alpha+\beta-2}\left(\frac{1}{2}s^2 B(z_+,z_+) + A((p,q))\right) \\
&= \sigma^{\alpha+\beta-2}\left(\frac{1}{2}s^2\langle Lz_+,z_+\rangle - \left\|(p,q)\right\|^2\right) \\
&= \sigma^{\alpha+\beta-2}\left(\frac{1}{2}s^2\langle \lambda z_+,z_+\rangle - \left\|(p,q)\right\|^2\right) \\
&= \sigma^{\alpha+\beta-2}\left(\frac{1}{2}s^2\lambda - \left\|(p,q)\right\|^2\right)
\end{aligned}
\tag{2.16}
$$

by (2.2), (2.4), (2.3), (2.5) and $z_+ = (p_+, q_+) \in E_+$ is a fixed eigenvector of L associated to an eigenvalue λ and $\|z_+\| = 1$. From hypothesis (H1) we see that for $s = 0$,

$$
J_K(z) \leqslant 0. \tag{2.17}
$$

Since $\int_0^T \sin\omega jt\,dt = 0$, $\int_0^T \cos\omega jt\,dt = 0$, $(p_+, q_+) \in E_+$, $(p,q) \in E_-$, and definition of E_+, E_- we obtain

$$
\int_0^T \left(\sigma^{\beta-1}(sp_+ + p) + p_0\right)dt = \int_0^T p_0\,dt = T\cdot p_0. \tag{2.18}
$$

It follows from (2.18) and Hölder inequality that

$$
\begin{aligned}
|p_0| &= \frac{1}{T}\left|\int_0^T \left(\sigma^{\beta-1}(sp_+ + p) + p_0\right)dt\right| \\
&\leqslant \frac{1}{T}\int_0^T \left|\sigma^{\beta-1}(sp_+ + p) + p_0\right|dt \\
&\leqslant \frac{1}{T}\left(\int_0^T \left|\sigma^{\beta-1}(sp_+ + p) + p_0\right|^\alpha dt\right)^{1/\alpha}\cdot\left(\int_0^T 1^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha} \\
&= (T)^{-1/\alpha}\left\|\sigma^{\beta-1}(sp_+ + p) + p_0\right\|_\alpha.
\end{aligned}
$$

Then, we have

$$
T|p_0|^\alpha \leqslant \left\|\sigma^{\beta-1}(sp_+ + p) + p_0\right\|_\alpha^\alpha. \tag{2.19}
$$

From (2.19) we obtain

$$\begin{aligned}\sigma^{(\beta-1)\alpha}\big\|(sp_+ + p)\big\|_\alpha^\alpha &= \int_0^T \big|\sigma^{\beta-1}(sp_+ + p)\big|^\alpha\,dt \\ &\leqslant 2^\alpha \int_0^T \big(\big|\sigma^{\beta-1}(sp_+ + p) + p_0\big|^\alpha + |p_0|^\alpha\big)\,dt \\ &= 2^\alpha\big(\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha + T|p_0|^\alpha\big) \\ &\leqslant 2^{\alpha+1}\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha. \end{aligned}\tag{2.20}$$

By (2.19) and (2.20), we have

$$\begin{aligned}&\sigma^{(\beta-1)\alpha}\big\|(sp_+ + p)\big\|_\alpha^\alpha + T|p_0|^\alpha \\ &\quad\leqslant 2^{\alpha+1}\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha + \big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha \\ &\quad\leqslant 2^{\alpha+2}\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha,\end{aligned}$$

i.e.,

$$\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha \geqslant \frac{1}{2^{\alpha+2}}\big(\sigma^{(\beta-1)\alpha}\big\|(sp_+ + p)\big\|_\alpha^\alpha + T|p_0|^\alpha\big).\tag{2.21}$$

Similarly, one has

$$\big\|\sigma^{\alpha-1}(sq_+ + q) + q_0\big\|_\beta^\beta \geqslant \frac{1}{2^{\beta+2}}\big(\sigma^{(\alpha-1)\beta}\big\|(sq_+ + q)\big\|_\beta^\beta + T|q_0|^\beta\big).\tag{2.22}$$

Taking $c_6 = c_1\min\{1/2^{\alpha+2}, 1/2^{\beta+2}\}$, for every point in Q we have from Lemma 2.1, (2.21) and (2.22),

$$\begin{aligned}\int_0^T H_K(z,t)\,dt &\geqslant c_1\big(\big\|\sigma^{\beta-1}(sp_+ + p) + p_0\big\|_\alpha^\alpha + \big\|\sigma^{\alpha-1}(sq_+ + q) + q_0\big\|_\beta^\beta\big) - Tc_2 \\ &\geqslant c_6\big(\sigma^{(\beta-1)\alpha}\|sp_+ + p\|_\alpha^\alpha + \sigma^{(\alpha-1)\beta}\|sq_+ + q\|_\beta^\beta \\ &\quad + T\big(|p_0|^\alpha + |q_0|^\beta\big)\big) - Tc_2.\end{aligned}\tag{2.23}$$

We assume now that $\alpha \leqslant \beta$ (the case $\alpha \geqslant \beta$ can be treated in an analogous way). By the triangle inequality we have

$$\|sp_+\|_\alpha \leqslant \frac{1}{2}\|sp_+ + p\|_\alpha + \frac{1}{2}\|sp_+ - p\|_\alpha.\tag{2.24}$$

From Lemma 2.3 and the fact that for all $z \in L^\gamma$, $\bar{\bar{z}} = -z$, one has for $(p,q) \in E_-$,

$$\bar{p} = q, \quad \bar{\bar{p}} = \bar{q}, \quad -p = \bar{q}.\tag{2.25}$$

Similarly, for $(p_+, q_+) \in E_+$ we obtain

$$p_+ = \bar{q}_+.\tag{2.26}$$

From (2.25), (2.26) and Marchel Riesz's theorem in Section 12.9 of [32], one obtains

$$\|sp_+ - p\|_\alpha = \|s\bar{q}_+ + \bar{q}\|_\alpha \leqslant C_0\|sq_+ + q\|_\alpha.\tag{2.27}$$

Using Hölder inequality, we have

$$\begin{aligned}\|sq_+ + q\|_\alpha^\alpha &= \int_0^T |sq_+ + q|^\alpha \cdot 1\,dt \\ &\leqslant \left(\int_0^T |sq_+ + q|^{\alpha\cdot(\beta/\alpha)}\,dt\right)^{\alpha/\beta} \cdot \left(\int_0^T 1^{\beta/(\beta-\alpha)}\,dt\right)^{(\beta-\alpha)/\beta} \\ &= \|sq_+ + q\|_\beta^\alpha \cdot T^{(\beta-\alpha)/\beta}. \end{aligned} \tag{2.28}$$

Taking $c_7 = \frac{1}{2}C_0 T^{(\beta-\alpha)/(\alpha\beta)}$, from (2.24), (2.27) and (2.28) we obtain

$$\|sp_+\|_\alpha \leqslant \frac{1}{2}\|sp_+ + p\|_\alpha + c_7\|sq_+ + q\|_\beta, \tag{2.29}$$

and then the existence of a constant $c > 0$ for all $s > 0$, $(p, q) \in E_-$ follows such that

$$\|sp_+ + p\|_\alpha \geqslant sc \tag{2.30}$$

or

$$\|sq_+ + q\|_\beta \geqslant sc. \tag{2.31}$$

Otherwise, for every $c > 0$, there exists $s > 0$ and $(p, q) \in E_-$ such that

$$\|sp_+ + p\|_\alpha \leqslant sc, \qquad |sq_+ + q\|_\beta \leqslant sc. \tag{2.32}$$

Then, from (2.29) and (2.32) we have

$$\|sp_+\|_\alpha \leqslant sc\left(\frac{1}{2} + c_4\right).$$

Thus, $\|p_+\|_\alpha \leqslant (1/2 + c_4)c$. Since c is random, we obtain $p_+ = 0$. And we have $q_+ = 0$ from Lemma 2.4. It contradicts that $\|(p_+, q_+)\| = 1$. In case that (2.30) holds, it follows from (2.16) and (2.23) that

$$J_K(z) \leqslant \frac{1}{2}\sigma^{\alpha+\beta-2}\lambda s^2 - c_6 c^\alpha \sigma^{(\beta-1)\alpha} s^\alpha + Tc_2. \tag{2.33}$$

In case (2.31) holds, it follows from (2.16) and (2.23) that

$$J_K(z) \leqslant \frac{1}{2}\sigma^{\alpha+\beta-2}\lambda s^2 - c_6 c^\beta \sigma^{(\alpha-1)\beta} s^\beta + Tc_2. \tag{2.34}$$

Choosing $s = \sigma$, and taking σ large enough it follows from $1/\alpha + 1/\beta < 1$, (2.33) and (2.34) that

$$J_K(z) \leqslant 0. \tag{2.35}$$

Finally we choose M. Given $s \in (0, \sigma)$ we have from (2.16) and (2.23) that

$$\begin{aligned} J_K(z) &\leqslant \frac{1}{2}\sigma^{\alpha+\beta-2}\lambda s^2 - \sigma^{\alpha+\beta-2}\big\|(p, q)\big\|^2 - Tc_6\big(|p_0|^\alpha + |q_0|^\beta\big) - Tc_2 \\ &\leqslant \frac{1}{2}\lambda\sigma^{\alpha+\beta} - \sigma^{\alpha+\beta-2}\big\|(p, q)\big\|^2 - Tc_6\big(|p_0|^\alpha + |q_0|^\beta\big) - Tc_2, \end{aligned}$$

so that if M is enough large and $\|(p,q)+z_0\| = M$ we have

$$J_K(z) \leqslant 0. \tag{2.36}$$

Thus, from (2.17), (2.35) and (2.36) we obtain

$$J_K(z) \leqslant 0 \quad \forall z \in \partial Q. \qquad \Box$$

Lemma 2.9. J_K *satisfies* (I1)–(I3).

Proof. From (2.1) and (2.2) we have

$$\begin{aligned} J_K(z) &= A(z) - \int_0^T H_K(z,t)\,dt = \frac{1}{2}B(z,z) - \int_0^T H_K(z,t)\,dt \\ &= \frac{1}{2}\langle Lz, z\rangle - \int_0^T H_K(z,t)\,dt. \end{aligned}$$

Taking $\Psi_K(z) = \int_0^T H_K(z,t)\,dt$, then for all $u \in E$ we obtain

$$\big(J_K'(z), u\big) = \langle Lz, u\rangle - \int_0^T \nabla H_K(z,t)\cdot u\,dt = \langle Lz, u\rangle - \big\langle \nabla\Psi_K(z), u\big\rangle.$$

So $\nabla J_K = L + (-\nabla\Psi_K)$, where L is a linear bounded selfadjoint operator and $(-\nabla\Psi_K)$ is a compact operator. Thus, J_K satisfies (I1) and (I2).

From the definition of L, one has

$$\langle Lz, \eta\rangle = \int_0^T p\cdot\dot\psi + \phi\cdot\dot q\,dt,$$

where $z = (p,q)$ and $\eta = (\phi,\psi)$ are elements in E. By analyzing in term of the Fourier series we easily obtain that

$$L(p,q) = (\bar q, -\bar p). \tag{2.37}$$

It is well known that

$$\begin{aligned} \exp(vL) &= 1 + vL + \frac{1}{2!}v^2L^2 + \frac{1}{3!}v^3L^3 + \frac{1}{4!}v^4L^4 + \cdots, \\ \cosh(vL) &= 1 + \frac{1}{2!}v^2L^2 + \frac{1}{4!}v^4L^4 + \cdots, \\ \sinh(vL) &= vL + \frac{1}{3!}v^3L^3 + \frac{1}{5!}v^5L^5 + \cdots. \end{aligned}$$

Thus, from (2.37) and recalling that $\bar{\bar p} = -p$ for every $p \in L^\gamma$ one obtains

$$\exp(vL)(p,0) = \cosh(v)(p,0) - \sinh(v)(0,\bar p) \tag{2.38}$$

and in a similar way

$$\exp(vL)(0,q) = \cosh(v)(0,q) - \sinh(v)(\bar{q},0). \tag{2.39}$$

Adding (2.38) and (2.39) we have

$$\exp(vL)(p,q) = \bigl(\cosh(v)p - \sinh(v)\bar{q}, \sinh(v)\bar{p} + \cosh(v)q\bigr). \tag{2.40}$$

We can give an explicit formula for $\hat{B}$. Given $z \in X$ we write $z = (p,q) + z_0$, where $(p,q) \in E_0$ and z_0. First we obtain a formula for $\hat{B}(p,q)$. From (2.8), (2.9), (2.40) and noting that $\bar{p} = q$ when $(p,q) \in E_-$ one has

$$B_1^{-1} \exp(vL) B_2(p,q) = (a_3 p, a_4 q), \tag{2.41}$$

where

$$a_3 = \frac{1}{\rho^{\beta-1}}\bigl(\cosh(v)\sigma^{\beta-1} - \sinh(v)\sigma^{\alpha-1}\bigr)$$

and

$$a_4 = \frac{1}{\rho^{\alpha-1}}\bigl(-\sinh(v)\sigma^{\beta-1} + \cosh(v)\sigma^{\alpha-1}\bigr).$$

Using the formula for the projection into E_- given in Lemma 2.5 and Lemma 2.3 from (2.41) we obtain

$$\begin{aligned}\hat{B}(p,q) = P_X(a_3 p, a_4 q) &= \frac{1}{2}(a_3 p, a_4 q) - \frac{1}{2}L(a_3 p, a_4 q)\\ &= \frac{1}{2}(a_3 p, a_4 q) + \frac{1}{2}(a_4 p, a_3 q) = \frac{1}{2}(a_3 + a_4)(p,q) := \frac{1}{2}a_5(p,q).\end{aligned}$$

An easy computation shows that

$$a_5 = \left(\frac{\sigma^{\beta-1}}{\rho^{\beta-1}} + \frac{\sigma^{\alpha-1}}{\rho^{\alpha-1}}\right)\cosh(v) - \left(\frac{\sigma^{\alpha-1}}{\rho^{\beta-1}} + \frac{\sigma^{\beta-1}}{\rho^{\alpha-1}}\right)\sinh(v).$$

If we assume $\sigma > 1$ and $\rho < 1$ it is easy to see that $a_5 > 0$. Then we have

$$\hat{B}\bigl((p,q) + z_0\bigr) = \frac{1}{2}a_5(p,q) + \left(\frac{1}{\rho^{\beta-1}}p_0, \frac{1}{\rho^{\alpha-1}}q_0\right)$$

so that $\hat{B}$ is invertible. Thus, (I3) is satisfied. □

Proof of Theorem 1.1. Using Lemmas 2.6–2.9, the hypothesis of Theorem 2.2 is satisfied. Thus a critical point z_K of J_K with critical value $c_K \geqslant \delta$ characterized by

$$c_K = \inf_{h\in\Gamma} \sup_{z\in Q} I\bigl(h(z,1)\bigr). \tag{2.42}$$

Next we show that as a consequence of hypothesis (H4) for K large enough $\|z_K\|_\infty < K$ so that z_K is a solution of (1.1). Recalling that the constants σ and M are independent of K, from (2.7) we obtain that Q is a bounded set. Thus, there is a constants $c_8 > 0$ satisfying that $\|z\| \leqslant c_8$ for every $z \in Q$. Then, from (H1), the definition of H_K, and the boundedness of Q, taking $h = id \in \Gamma$ from (2.42) we obtain

$$c_K \leqslant \sup_{z\in Q} J_K(z) \leqslant \frac{1}{2}\langle Lz, z\rangle \leqslant \frac{1}{2}c_8\|L\|.$$

Denoting $z_K = (p_K, q_K)$ we have

$$\begin{aligned}\frac{1}{2}c_8\|L\| \geqslant c_K = J_K(z_K) &= J_K(z_K) - J_K'(z_K)\cdot p_K \\ &= -\int_0^T \big(H_K(z_K,t) - H_K p(z_K,t)\cdot p_K\big)\,dt. \qquad (2.43)\end{aligned}$$

Similarly, one has

$$\frac{1}{2}c_8\|L\| \geqslant -\int_0^T \big(H_K(z_K,t) - H_K q(z_K,t)\cdot q_K\big)\,dt. \qquad (2.44)$$

Multiplying (2.43) by $1/\alpha$, (2.44) by $1/\beta$, and adding we obtain

$$\begin{aligned}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)c_8\|L\| \geqslant -\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\int_0^T H_K(z_K,t)\,dt \\ + \int_0^T \frac{1}{\alpha}H_{Kp}(z,t)\cdot p_K + \frac{1}{\beta}H_{Kq}(z,t)\cdot q_K\,dt. \qquad (2.45)\end{aligned}$$

Then, using hypothesis (H2) that is also satisfied by H_K with the same constants

$$\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\|L\|c_8 \geqslant \left(1-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\int_0^T H_K(z_K,t)\,dt. \qquad (2.46)$$

Using Lemma 2.1 from (2.46) we obtain that for a constant c_9 independent of K,

$$\|p_K\|_\alpha^\alpha + \|q_K\|_\beta^\beta \leqslant c_9. \qquad (2.47)$$

Since z_K satisfies (1.1) we have from hypothesis (H4) and (2.45) that

$$\begin{aligned}\|\dot z_K\|_1 &= \int_0^T \big|H_{Kz}(z_K,t)\big|\,dt \\ &\leqslant a\int_0^T \left(\frac{1}{\alpha}H_{Kp}(z_K,t)\cdot p_K + \frac{1}{\beta}H_{Kq}(z_K,t)\cdot q_K\right)dt + Tb \leqslant c_{10} \qquad (2.48)\end{aligned}$$

with c_{10} independent of K. Then (2.47) and (2.48) give that $\|z_K\|_\infty$ is bounded independent of K, which finishes the proof. □

Proof of Theorem 1.2. We can take $T = 2\pi$ and choose $k \in \mathcal{N}$. It is convenient to make the change of variables $\tau = k^{-1}t$. Thus $z(t)$ is a $2k\pi$ periodic solution of (1.1), $\zeta(\tau) = z(k\tau)$ satisfies

$$\frac{d\zeta}{d\tau} = k\mathcal{J}H_z(\zeta, k\tau), \tag{2.49}$$

by

$$\frac{d\zeta(\tau)}{d\tau} = k \cdot \frac{dz(k\tau)}{dk\tau} = k\mathcal{J}H_z\big(z(k\tau), k\tau\big) = k\mathcal{J}H_z\big(\zeta(\tau), k\tau\big).$$

We seek a 2π periodic solution of (2.49). Since $kH(z, k\tau)$ satisfies (H0)–(H4), Theorem 1.1 provides a critical point $\zeta_{k,K}(\tau) \in E$ of

$$J_{k,K}(\zeta) = A(\zeta) - k\int_0^{2\pi} H_K(\zeta, k\tau)\,dt,$$

where K depends on k, which for appropriately large K is a classical solution of (2.49).

Let $\zeta_k(\tau)$ is a classical solution of (2.49). Note that $\zeta_1(k\tau)$ also satisfies (2.49) by

$$\frac{d\zeta_1(k\tau)}{d\tau} = k \cdot \frac{d\zeta_1(k\tau)}{dk\tau} = k \cdot \mathcal{J}H_z\big(\zeta_1(k\tau), k\tau\big).$$

Taking

$$J_k(\zeta) = A(\zeta) - k\int_0^{2\pi} H\big(\zeta(\tau), k\tau\big)\,d\tau,$$

if $\zeta_1(k\tau) = \zeta_k(\tau)$, we have

$$\begin{aligned}
J_k(\zeta_k) &= A\big(\zeta_k(\tau)\big) - k\int_0^{2\pi} H\big(\zeta_k(\tau), k\tau\big)\,d\tau \\
&= \int_0^{2\pi} p_k(\tau) \cdot \frac{dq_k(\tau)}{d\tau}\,dt - k\int_0^{2\pi} H\big(\zeta_k(\tau), k\tau\big)\,d\tau \\
&= k\int_0^{2\pi} \left(p_1(k\tau) \cdot \frac{dq_1(k\tau)}{dk\tau} - H\big(\zeta_1(k\tau), k\tau\big)\right)d\tau \\
&= \int_0^{2k\pi} \left(p_1(s) \cdot \frac{dq_1(s)}{ds} - H(\zeta_1(s), s)\right)ds = kJ_1(\zeta_1).
\end{aligned}$$

Thus, one has

$$d_k \equiv J_k(\zeta_k) = kJ_1(\zeta_1) = kd_1.$$

Since $d_1 > 0$ by Theorem 2.2, it follows that $d_k \to \infty$ as $k \to \infty$. We shall show next that this is impossible since d_k is bounded from above independently of k. Recall from Theorem 2.2, (2.16) and (H1) that

$$\begin{aligned} d_k &\leqslant \sup_{z\in Q} J_{k,K}(z) \\ &= \sup_{0\leqslant s\leqslant\sigma,\, 0\leqslant\|(p,q)+z_0\|\leqslant M} \left(\sigma^{\alpha+\beta-2}\left(\frac{1}{2}\lambda s^2 - \|(p,q)\|^2\right) - k\int_0^{2\pi} H_K(z,k\tau)\,d\tau\right) \\ &\leqslant \lambda\big(\sigma(k)\big)^{\alpha+\beta}, \end{aligned}$$

where we have written $\sigma(k)$ to emphasize its dependence on k. The parameter $\sigma(k)$ was determined in (2.33) and (2.34) when $s=\sigma$. The corresponding equation satisfied by $\sigma(k)$ is

$$\frac{1}{2}\lambda\sigma^{\alpha+\beta} - kc_6c^{\alpha}\sigma^{\alpha\beta} + 2k\pi c_2 \leqslant 0$$

or

$$\frac{1}{2}\lambda\sigma^{\alpha+\beta} - kc_6c^{\beta}\sigma^{\alpha\beta} + 2k\pi c_2 \leqslant 0$$

for all $\sigma \geqslant \sigma(k)$. It follows that

$$\sigma(k) = \max\left\{\left(\frac{\lambda}{kc_6c^{\alpha}}\right)^{1/(\alpha\beta-(\alpha+\beta))}, \left(\frac{4\pi c_2}{c_6c^{\alpha}}\right)^{1/(\alpha\beta)}\right\}$$

or

$$\sigma(k) = \max\left\{\left(\frac{\lambda}{kc_6c^{\beta}}\right)^{1/(\alpha\beta-(\alpha+\beta))}, \left(\frac{4\pi c_2}{c_6c^{\beta}}\right)^{1/(\alpha\beta)}\right\},$$

which implies that $\{\sigma(k) \mid k\in N\}$ is bounded. Thus the critical values d_k are bounded and therefore there is k_1 such that $\zeta_1(k\tau) \neq \zeta_k(\tau)$ for all $k \geqslant k_1$. Reapplying what we have just shown to the 2π periodic function $k_1H(z,k_1\tau)$ it follows that there is a sequence of nonzero 2π periodic solutions $z_j(\tau)$ to

$$\frac{dz}{d\tau} = jk_1\mathcal{J}H_z(z, jk_1\tau) \tag{2.50}$$

with $z_j(\tau) \neq z_1(j\tau)$ for all $j \geqslant k_2$. Moreover, from the form of (2.50) and the corresponding variational problem, $z_j(\tau) = \zeta_{jk_1}(\tau)$ and $z_j(\tau) \neq \zeta_1(jk_1\tau)$ for all $j \geqslant k_2$. It follows that we have a sequence

$$\zeta_1(t), \zeta_{k_1}\left(\frac{t}{k_1}\right), \zeta_{k_1k_2}\left(\frac{t}{k_1k_2}\right), \ldots,$$

of distinct nonzero solutions of (1.1) and the proof is complete. □

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